

# IMPLICATIONAL COMPLETENESS

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ABSTRACT. For the implicational propositional calculus, we present a proof of completeness based on a variant of the Lindenbaum procedure.

## 0. INTRODUCTION

A standard formulation of the Implicational Propositional Calculus (IPC) has  $\supset$  as its only connective, has modus ponens as its only inference rule, and has the following axiom schemes:

$$\begin{aligned} A &\supset (B \supset A) \\ [A &\supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)] \\ [(A &\supset B) \supset A] &\supset A \end{aligned}$$

the last of which is due to Peirce. A (Boolean) *valuation* is a map  $v$  from the set  $wf$  of all well-formed formulas to the set  $\{0,1\}$  such that  $v(A \supset B) = 0$  precisely when  $v(A) = 1$  and  $v(B) = 0$ ; a *tautology* is a well-formed formula that takes the value 1 in all such valuations.

The *Completeness Theorem* for IPC asserts that every tautology is a theorem: if a well-formed formula has value 1 in each valuation then there is a proof of it using modus ponens and the three axiom schemes listed above. One proof of completeness for IPC is indicated in Exercises 6.3, 6.4, 6.5 of Robbin [2]; that proof adapts the Kalm  r approach for the classical propositional calculus. Our purpose in this brief paper is to present a proof of completeness for IPC that adapts the Lindenbaum approach for the classical propositional calculus.

## 1. THEOREM AND PROOF

We begin with some simple observations regarding IPC. To say that  $A \in wf$  may be deduced from  $\Gamma \subseteq wf$  we may write  $\Gamma \vdash A$  as usual; in particular (taking  $\Gamma$  to be empty)  $\vdash A$  asserts that  $A$  is a theorem. Modus ponens and the first two axiom schemes together ensure that if  $A \in wf$  is any well-formed formula then  $A \supset A$  is a theorem. Modus ponens and the first two axiom schemes further ensure that the *Deduction Theorem* (DT) holds: if  $\Gamma \cup \{A\} \vdash B$  then  $\Gamma \vdash A \supset B$ ; in particular, if  $A \vdash B$  then  $\vdash A \supset B$ .

The lack of negation in IPC is in part repaired by an elegant device. Fix an arbitrary well-formed formula  $Q \in wf$ . When  $A \in wf$  write  $QA := Q(A) := A \supset Q$ .

**Theorem 1.** *Fix any well-formed formula  $Q \in wf$ . If  $A, B, C \in wf$  then each of the following well-formed formulas is a theorem of IPC:*

- (1)  $(A \supset B) \supset [(B \supset C) \supset (A \supset C)]$
- (2)  $(A \supset B) \supset (QB \supset QA)$
- (3)  $A \supset QQA$
- (4)  $QQQA \supset QA$
- (5)  $QQB \supset QQ(A \supset B)$
- (6)  $QQA \supset [QB \supset Q(A \supset B)]$
- (7)  $QA \supset QQ(A \supset B)$
- (8)  $(QA \supset B) \supset [(QQA \supset B) \supset QQB]$ .

*Proof.* This is Exercise 6.3 in Chapter 1 of Robbin [2]. As noted by Robbin, part (7) requires the Peirce axiom scheme; the other parts need only the first two axiom schemes.  $\square$

The classical propositional calculus presented by Church [1] and followed by Robbin [2] incorporates a propositional symbol  $\mathbf{f}$  (falsity) having value 0 under each valuation; it takes  $\mathbf{f}A = A \supset \mathbf{f}$  for the negation  $\sim A$  of  $A$ ; and it replaces the Peirce axiom scheme by the ‘double negation’ axiom scheme  $\sim\sim A \supset A$ . The symbol  $\mathbf{f}$  has no place in the present paper; however, significant aspects of its function will be served by a non-theorem  $Q$ .

From this point on, we shall consider extensions of IPC obtained by enlarging the set of theorems, so we shall modify our notation accordingly. Let us write  $L$  for the system IPC as formulated above and write  $\mathbb{T}(L) = \{A \in wf : \vdash A\}$  for its set of theorems. An extension  $M$  of  $L$  is produced by adding an axiom (or axioms); thus  $\mathbb{T}(L) \subseteq \mathbb{T}(M)$  and the Deduction Theorem continues to hold for  $M$ . To indicate that a well-formed formula  $A$  is a theorem of  $M$  we prefer to write  $A \in \mathbb{T}(M)$  rather than the customary  $\vdash_M A$ .

**Theorem 2.** *Let  $Q \in wf$  and let  $M$  be an extension of  $L$ . The following are equivalent:*

- (1)  $Q \in \mathbb{T}(M)$ ;
- (2) *some  $A \in wf$  has  $A \in \mathbb{T}(M)$  and  $QA \in \mathbb{T}(M)$ ;*
- (3) *some  $A \in wf$  has  $QQA \in \mathbb{T}(M)$  and  $QA \in \mathbb{T}(M)$ .*

*Proof.* (2)  $\Rightarrow$  (3) Follows from Theorem 1 part (3) by modus ponens.

(3)  $\Rightarrow$  (1) Follows by modus ponens from  $\mathbb{T}(M) \ni QA$  and  $\mathbb{T}(M) \ni QQA (= QA \supset Q)$ .

(1)  $\Rightarrow$  (2) Simply let  $A = Q$  and recall that  $QQ (= Q \supset Q) \in \mathbb{T}(L) \subseteq \mathbb{T}(M)$ .  $\square$

We say that  $M$  is *Q-inconsistent* precisely when it satisfies one (hence each) of the equivalent conditions in this theorem; we say that  $M$  is *Q-consistent* otherwise.

Henceforth, we shall let  $Q \in wf$  be a well-formed formula that is **not** a theorem of  $L$ ; thus,  $L$  is *Q-consistent*.

**Theorem 3.** *Let  $M$  be a Q-consistent extension of  $L$  and  $A \in wf$  a well-formed formula. If  $QQA$  is not a theorem of  $M$ , then the extension  $N$  of  $M$  obtained by adding  $QA$  as an axiom is Q-consistent.*

*Proof.* To prove the contrapositive, assume that  $Q \in \mathbb{T}(N)$ : a deduction of  $Q$  within the system  $N$  is a deduction of  $Q$  from  $QA$  within the system  $M$ ; by the Deduction Theorem, it follows that  $QQA = (QA \supset Q) \in \mathbb{T}(M)$ .  $\square$

We say that  $M$  is *Q-complete* precisely when each  $A \in wf$  satisfies either  $QQA \in \mathbb{T}(M)$  or  $QA \in \mathbb{T}(M)$ ; that is, either  $QQA$  or  $QA$  is a theorem of  $M$ .

**Theorem 4.** *Each Q-consistent extension  $M$  of  $L$  has a Q-complete Q-consistent extension.*

*Proof.* List all the well-formed formulas: say  $wf = \{A_n : n \geq 0\} = \{A_0, A_1, \dots\}$ .

Put  $N_0 = M$ . If  $QQA_0 \in \mathbb{T}(N_0)$  then let  $N_1 = N_0$ ; if  $QQA_0 \notin \mathbb{T}(N_0)$  then let  $N_1$  be  $N_0$  with  $QA_0$  as an extra axiom. Repeat inductively: if  $QQA_n \in \mathbb{T}(N_n)$  then let  $N_{n+1} = N_n$ ; if  $QQA_n \notin \mathbb{T}(N_n)$  then let  $N_{n+1}$  be  $N_n$  with  $QA_n$  as an extra axiom. Finally, let  $N$  be the extension of  $M$  produced by adding as axioms all those  $wfs$  introduced at each stage of this inductive process.

Claim:  $N$  is Q-consistent. [Any proof of  $Q$  in  $N$  would involve only finitely many axioms and would therefore be a proof of  $Q$  in  $N_n$  for some  $n \geq 0$ ; but Theorem 3 guarantees inductively that the extension  $N_n$  is Q-consistent for each  $n \geq 0$ .]

Claim:  $N$  is Q-complete. [Take any  $wf$ : say  $A_n$ . If  $QQA_n \in \mathbb{T}(N_n)$  then  $QQA_n \in \mathbb{T}(N)$ ; if  $QQA_n \notin \mathbb{T}(N_n)$  then  $QA_n \in \mathbb{T}(N_{n+1})$  so that  $QA_n \in \mathbb{T}(N)$ . Thus, if  $A \in wf$  is arbitrary then either  $QQA$  or  $QA$  is a theorem of  $N$  as required.]  $\square$

Now, let  $M$  be a  $Q$ -consistent extension of  $L$  and let  $N$  be a  $Q$ -complete  $Q$ -consistent extension of  $M$ . Let  $A$  be a well-formed formula: when  $QQA$  is a theorem of  $N$  we put  $v_N(A) = 1$ ; when  $QA$  is a theorem of  $N$  we put  $v_N(A) = 0$ . As  $N$  is both  $Q$ -complete and  $Q$ -consistent, this defines a function  $v_N : wf \rightarrow \{0, 1\}$ .

**Claim:**  $v = v_N$  is a valuation: that is,  $v(A \supset B) = 0$  precisely when  $v(A) = 1$  and  $v(B) = 0$ .

*Proof:* Suppose that  $v(A) = 0$ : that is, suppose  $QA \in \mathbb{T}(N)$ ; Theorem 1 part (7) tells us that

$$QA \supset QQ(A \supset B) \in \mathbb{T}(L) \subseteq \mathbb{T}(N)$$

whence modus ponens places  $QQ(A \supset B)$  in  $\mathbb{T}(N)$  and  $v(A \supset B) = 1$ . Suppose  $v(B) = 1$ : that is, suppose  $QQB \in \mathbb{T}(N)$ ; Theorem 1 part (5) tells us that

$$QQB \supset QQ(A \supset B) \in \mathbb{T}(L) \subseteq \mathbb{T}(N)$$

whence modus ponens places  $QQ(A \supset B)$  in  $\mathbb{T}(N)$  and  $v(A \supset B) = 1$ . Thus

$$(v(A) = 0) \vee (v(B) = 1) \Rightarrow v(A \supset B) = 1$$

and so

$$v(A \supset B) = 0 \Rightarrow (v(A) = 1) \wedge (v(B) = 0).$$

Conversely, let  $v(A) = 1$  and  $v(B) = 0$ : thus,  $QQA$  and  $QB$  are theorems of  $N$ ; part (6) of Theorem 1 tells us that

$$QQA \supset [QB \supset Q(A \supset B)] \in \mathbb{T}(L) \subseteq \mathbb{T}(N)$$

whence two applications of modus ponens yield  $Q(A \supset B) \in \mathbb{T}(N)$  and so  $v(A \supset B) = 0$ . □

We are now able to prove the completeness of IPC.

**Theorem 5.** *The Implicational Propositional Calculus is complete.*

*Proof.* Suppose that  $Q$  is not a theorem of  $L$ ; thus,  $L$  is  $Q$ -consistent. Theorem 4 fashions a  $Q$ -complete  $Q$ -consistent extension  $N$  of  $L$  by means of which we define the valuation  $v_N$  as above. Before stating Theorem 1 we noted that  $Q \supset Q$  is a theorem of  $L$ ; thus  $QQ (= Q \supset Q)$  is a theorem of  $N$  and so  $v_N(Q) = 0$ . We have found a valuation under which  $Q$  does not take the value 1;  $Q$  is not a tautology. □

#### REFERENCES

- [1] Alonzo Church, *Introduction to Mathematical Logic*, Princeton University Press (1956).
- [2] Joel W. Robbin, *Mathematical Logic - A First Course*, W.A. Benjamin (1969); Dover Publications (2006).

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